

Maxwell-Higgs self-dual solitons on an infinite cylinder

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We have studied the Maxwell-Higgs model on the surface of an infinite cylinder. In particular we show that this model supports self-dual topological soliton solutions on the infinite tube. Finally, the Bogomol'nyi-type equations are studied from theoretical and numerical point of view.

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It is well known, that the abelian Higgs models in (2+1)-dimension with the Maxwell term present topological stable vortex solution [1] (for review see [2–5]). These models have the particularity to become auto-dual when the self-interactions are suitably chosen. When this occurs the model presents particular mathematical and physics properties, such as the supersymmetric extension of the model [6], and the reduction of the motion equation to first order derivative equation [7, 8]. Characteristically, these vortex solutions carry magnetic flux but are electrically neutral. They correspond to vortex-like objects carrying a magnetic field in its core, where the scalar field vanishes. Also, the magnetic flux is quantized.

Recently, there has been interest in the study of gauge theories on infinite tubes [9]. Particularly, it was studied $SU(2)$ Yang-Mills model within closed and open tubes, showing the existence of magnetic monopoles and dyons living within this tube-shaped domain.

Here we are interested in the study of the abelian Maxwell-Higgs model within an infinite cylinder. Our goal is to show that the abelian Maxwell-Higgs model supports topological solitons when defined within an infinite cylinder. Specifically, we propose an ansatz such that the model is reduced to a theory within an infinite cylinder surface. We will show that for the model defined in the cylinder it is possible to obtain self-dual or Bogomolnyi equations by minimizing the energy functional of the model. The model is then bounded below by topological number. Finally we analyze the numerical solutions of the field equations showing the behavior of matter and gauge fields as well as their energy density. We also analyze the behavior of the magnetic vortex.

We start by considering the Maxwell-Higgs electrodynamics in a (2+1)-dimensional curved manifold described by the following metric

$$ds^2 = dt^2 - \rho^2 d\phi^2 - dz^2, \quad (1)$$

(for solitons on curved manifold see [10–13]) where the azimuthal angle runs over $0 \leq \phi \leq 2\pi$, and the z -

coordinate varies between $0 \leq z_+ < +\infty$. Here, ρ is a constant and in the following we set $\rho = 1$, for simplicity. Such a manifold is a semi-infinite cylinder immersed in a (3+1)-dimensional space-time. The action describing the Maxwell electrodynamics coupled to complex scalar field $\psi(x)$ is

$$S = \int d^3x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \psi|^2 - U(|\psi|) \right], \quad (2)$$

The covariant derivative is defined as $D_\mu = \partial_\mu + ieA_\mu$ and the metric tensor is $g^{\mu\nu} = (1, -1, -1)$.

By varying with respect to A_μ and ψ^\dagger , we obtain the field equations

$$\partial^\mu F_{\mu\nu} + J_\nu = 0, \quad (3)$$

$$D_\mu D_\mu \psi + \frac{\partial U}{\partial \psi^*} = 0, \quad (4)$$

where $J_\nu = -ie[\psi^*(D_\nu \psi) - \psi(D_\nu \psi)^*]$.

The energy associated with the action (2) is

$$E = \int d^3x \left[\frac{1}{2} F_{0i}^2 + \frac{1}{4} F_{ji}^2 + |D_0 \psi|^2 + |D_i \psi|^2 + U(|\psi|) \right]. \quad (5)$$

From (3), the stationary Gauss law reads

$$\partial_j \partial_j A_0 + 2e^2 A_0 |\psi|^2 = 0, \quad (6)$$

it is satisfied identically by the configuration $A_0 = 0$, thus, in the static regime the model describes pure magnetic configurations.

In this note we are interested in the particular case of the static field configurations with $A_0 = 0$. In this context, the energy in (5) reduces to:

$$E = \int d^2x \left[\frac{1}{2} \mathbf{B}^2 + |D_i \psi|^2 + U(|\psi|) \right], \quad (7)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$.

Let us consider the following ansatz:

$$A_{z_+} = 0, \quad A_\phi = \frac{h(z_+)}{e}, \quad \psi = v f(z_+) e^{-in\phi}, \quad (8)$$

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where n is a topological number called the winding number. With such conditions we see that the magnetic field restricted to the cylinder surface reduce to:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\partial_{z_+} A_\phi \hat{\rho} = -\frac{\partial_{z_+} h(z_+)}{e} \hat{\rho}, \quad (9)$$

In addition the ϕ -component of the covariant derivative takes the form,

$$D_\phi \psi = \partial_\phi \psi + ie A_\phi \psi, \quad (10)$$

where we have used $\rho = 1$. Using the ansatz (8) we can replace ψ and A_ϕ , so that,

$$\begin{aligned} D_\phi \psi &= -invf(z_+)e^{-in\phi} + ivf(z_+)h(z_+)e^{-in\phi} \\ &= ivf(z_+) [-n + h(z_+)] e^{-in\phi}. \end{aligned} \quad (11)$$

Thus, we may develop the term $|D_\phi \psi|^2$ to be,

$$|D_\phi \psi|^2 = v^2 f(z_+)^2 [h(z_+) - n]^2, \quad (12)$$

On the other hand, the component $\hat{\mathbf{z}}_+$ of the covariant derivative reduce to

$$D_{z_+} \psi = \partial_{z_+} \psi = ve^{-in\phi} \partial_{z_+} f(z_+), \quad (13)$$

so that,

$$|D_{z_+} \psi|^2 = v^2 [\partial_{z_+} f(z_+)]^2. \quad (14)$$

Thus, it is possible, in order to establish the suitable boundary conditions, rewrite the energy (7) in terms of the ansatz (8)

$$\begin{aligned} E &= \int dz_+ d\phi \left[\frac{1}{2} \left(\frac{\partial_{z_+} h(z_+)}{e} \right)^2 + U(f) \right. \\ &\quad \left. + v^2 [\partial_{z_+} f(z_+)]^2 + v^2 f(z_+)^2 [h(z_+) - n]^2 \right], \end{aligned} \quad (15)$$

The field equations (3) and (4) may also be written in terms of the ansatz. The set of equations (3) reduce to one equation:

$$\partial_{z_+}^2 h(z_+) = 2v^2 e^2 [f(z_+)]^2 [h(z_+) - n], \quad (16)$$

whereas the equation (4) reads as:

$$\partial_{z_+}^2 f(z_+) - f(z_+) [h(z_+) - n]^2 - \frac{1}{2v} \frac{\partial U}{\partial f} = 0. \quad (17)$$

Appropriate boundary conditions for a soliton solution of finite energy should take the form

$$f(+\infty) = 1, \quad f(0) = \gamma_0^{(n)}, \quad (18)$$

$$h(+\infty) = n, \quad h(0) = 0, \quad (19)$$

where $\gamma_0^{(n)}$ is related to the winding number and therefore is uniquely determined for each topological sector. In

particular we will show by numerical analysis that, $0 < \gamma_0^{(n)} < 1$. Note, that this requirement, does not affect the regularity of the field $\psi(\phi, z)$ at $z_+ = 0$, since $z_+ = 0$ is a circle S_1 and then $\psi(\phi, 0)$ is a map $S_1 \rightarrow S_1$, i.e. for each angle ϕ we have a complex number $v\gamma_0^{(n)} e^{-in\phi}$.

The boundary conditions (19) allow to compute the magnetic flux, the important observable for the solitons described by the Maxwell-Higgs electrodynamics. So, by using the formulas (9) and (19), the magnetic flux reads

$$\begin{aligned} \Phi &= \int dz_+ d\phi B_\rho \\ &= 2\pi \int_0^\infty dz_+ \left(-\frac{\partial_{z_+} h(z_+)}{e} \right) = -\frac{2\pi}{e} n, \end{aligned} \quad (20)$$

as expected, it becomes a quantized quantity, i.e., proportional to the winding number n describing the respective topological sector.

In order to find the static field configurations that are stationary points of the energy, it is convenient to rewrite the expression of the energy as

$$\begin{aligned} E &= 2\pi \int_0^\infty dz_+ \left[\frac{1}{2} \left(\frac{\partial_{z_+} h(z_+)}{e} \pm \sqrt{2U} \right)^2 \right. \\ &\quad \left. + v^2 \left(\partial_{z_+} f(z_+) \pm f(z_+) [h(z_+) - n] \right)^2 \right. \\ &\quad \left. \mp \frac{\partial_{z_+} h(z_+)}{e} \sqrt{2U} \mp 2v^2 [h(z_+) - n] f(z_+) \partial_{z_+} f(z_+) \right]. \end{aligned} \quad (21)$$

Let us now define the the potential term $U(f)$ to be the usual symmetry breaking one of the Maxwell-Higgs model in its self-dual form,

$$U(f) = \frac{e^2 v^4}{2} (f^2 - 1)^2. \quad (22)$$

Now it is not difficult to see that

$$\partial_{z_+} \left(\frac{\sqrt{2U}}{e} \right) = 2v^2 f(z_+) \partial_{z_+} f(z_+), \quad (23)$$

which allow to rewrite the third line of Eq. (21) in the following way

$$\begin{aligned} &\mp \frac{\partial_{z_+} h(z_+)}{e} \sqrt{2U} \mp 2v^2 [h(z_+) - n] f(z_+) \partial_{z_+} f(z_+) \\ &= \mp \frac{1}{e} \partial_{z_+} \left([h(z_+) - n] \sqrt{2U} \right). \end{aligned} \quad (24)$$

Therefore, the energy (21) may be rewritten as a sum of squared terms plus a total derivative,

$$\begin{aligned} E &= 2\pi \int_0^\infty dz_+ \left[\frac{1}{2} \left(\frac{\partial_{z_+} h(z_+)}{e} \pm \sqrt{2U} \right)^2 \right. \\ &\quad \left. + v^2 \left(\partial_{z_+} f(z_+) \pm f(z_+) [h(z_+) - n] \right)^2 \right. \\ &\quad \left. \mp \frac{1}{e} \partial_{z_+} \left((h(z_+) - n) \sqrt{2U} \right) \right]. \end{aligned} \quad (25)$$

The total derivative may be explicitly evaluated by using the expression (22) and the boundary conditions (18) and (19). In such case we have,

$$\begin{aligned} & \mp 2\pi v^2 \int_0^\infty dz_+ \partial_{z_+} \left[(h(z_+) - n) (f(z_+)^2 - 1) \right] \\ & = \pm 2\pi v^2 n \left[1 - \left(\gamma_0^{(n)} \right)^2 \right] \end{aligned} \quad (26)$$

Therefore, we see that the energy is bounded below by a multiple of the magnitude of the winding number (for positive $n[1 - \gamma_0^2]$ we choose the upper sign, and for negative $n[1 - \gamma_0^2]$ we choose the lower sign). Here, it is interesting to note that the topological bound (26) consist on two terms. The first of these terms is the topological bound of the energy associated to the $(2+1)$ planar Maxwell-Higgs model, i.e.

$$\pm 2\pi v^2 n, \quad (27)$$

which is proportional to the magnetic flux (20). The second term is

$$\mp 2\pi v^2 n \left(\gamma_0^{(n)} \right)^2 \quad (28)$$

and it is a novel term that emerges in our theory. As we mention $\gamma_0^{(n)}$ depends on the winding number n , so that the topological bound (26) is unequivocally determined for each topological sector. In particular we will show, by numerical analysis, that $\gamma_0^{(n)} \rightarrow 0$ for large values of n , such that

$$\pm 2\pi v^2 n \left[1 - \left(\gamma_0^{(n)} \right)^2 \right] \rightarrow \pm 2\pi v^2 n \quad (29)$$

Thus, our topological bound tends, for large values of n , to the topological bound (27) of the Maxwell-Higgs model.

The bound (26) is saturated by fields satisfying the first-order Bogomol'nyi self-duality equations,

$$\frac{\partial_{z_+} h(z_+)}{e} \pm ev^2 (f^2 - 1) = 0 \quad (30)$$

$$\partial_{z_+} f(z_+) \pm f(z_+) (h(z_+) - n) = 0 \quad (31)$$

It easily verified that BPS equations (31) and (30) solve the second-order equations of motion given in (16) and (17) if the potential is given by (22).

On the other hand, by using self dual equations, the BPS energy density $\varepsilon(z_+)$, $[E = 2\pi\rho_0 \int_0^\infty dz_+ \varepsilon(z_+)]$, is expressed as

$$\varepsilon(z_+) = e^2 v^4 (f^2 - 1)^2 + 2v^2 f^2 (h - n)^2, \quad (32)$$

which is a positive-definite quantity.

Let us now concentrate in the solutions of the equations (31) and (30) at $z_+ \rightarrow +\infty$. From the preceding

arguments it is not difficult to find their asymptotic behavior

$$\begin{aligned} f(z_+) & \simeq 1 - \lambda_1 e^{-ev\sqrt{2}z_+} \\ h(z_+) & \simeq n - \lambda_1 ev\sqrt{2} e^{-ev\sqrt{2}z_+}, \end{aligned} \quad (33)$$

with λ_1 a real number. From these solutions we may evaluate the magnetic field at $z_+ \rightarrow +\infty$. By substituting $h(z_+)$ in the expression (9) for the magnetic field we have

$$B|_{z_+ \rightarrow \infty} = -2ev^2 \lambda_1 e^{-ev\sqrt{2}z_+}, \quad (34)$$

which shows the asymptotic behavior of the magnetic field.

We also analyze the behavior of the solution at $z_+ \rightarrow 0$. In such case we have for $n > 0$:

$$f(z_+) = \gamma_0^{(n)} - n\gamma_0^{(n)} z_+ + \dots, \quad (35)$$

$$h(z_+) = e^2 v^2 \left[1 - \left(\gamma_0^{(n)} \right)^2 \right] z_+ - e^2 v^2 \left(\gamma_0^{(n)} \right)^2 n z_+^2 + \dots,$$

where γ_0 is a real number depending on the value of n and it is determined numerically. Finally, we can evaluate the magnetic field near $z_+ = 0$ by substituting the last equation of (35) in the expression (9),

$$B|_{z_+ \rightarrow 0} = -ev^2 \left[1 - \left(\gamma_0^{(n)} \right)^2 \right]. \quad (36)$$

By comparing the expression (36) with (34) we see that the absolute value of the magnetic field has a maximum at $z_+ = 0$ and decrease exponentially as $z_+ \rightarrow +\infty$.

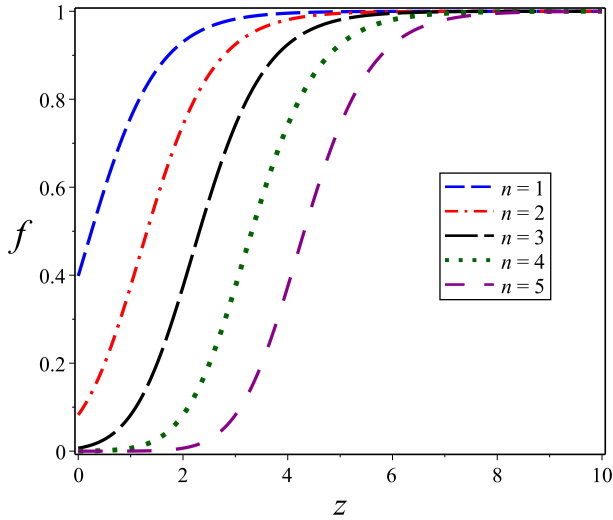
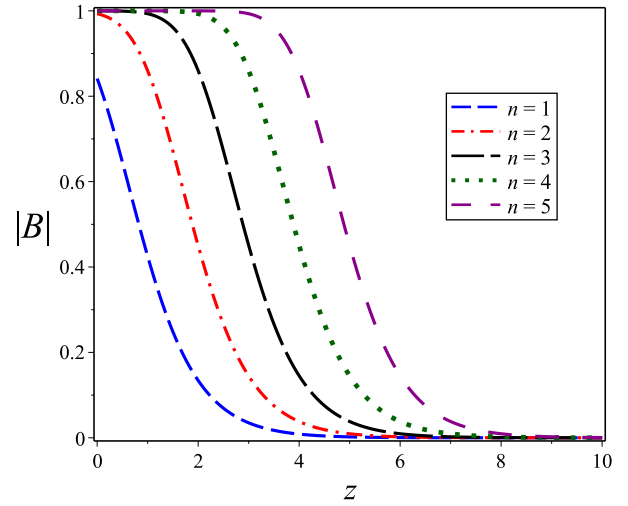
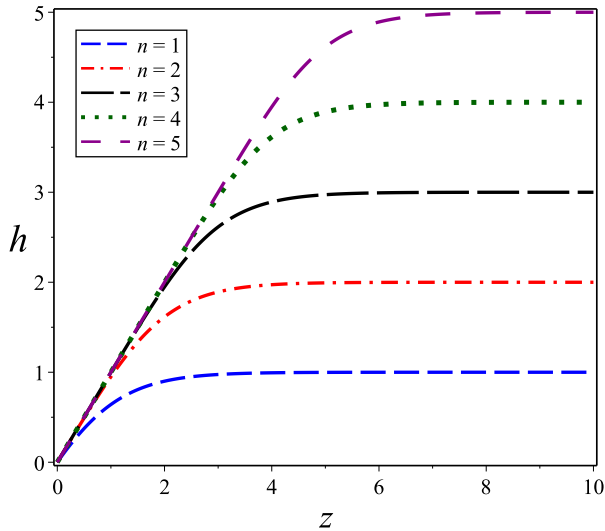
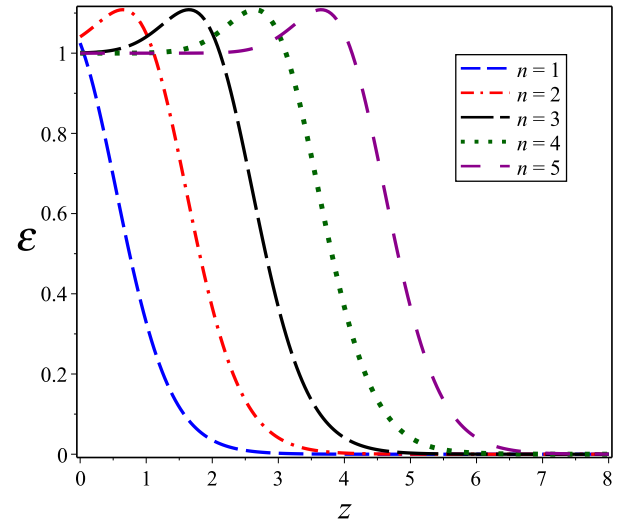
We have verified that the self dual equations solved with boundary conditions (18) and (19) provide well-behaved solutions both at origin as at infinity.

Below we show the numerical solutions of the self-dual equations (30) and (31) with boundary conditions given by (18) and (19) for some values of $n > 0$. To perform the numerical analysis we have considered $e = v = \rho_0 = 1$ and $n = 1, 2, 3, 4, 5$.

The Fig.1 depicts the profiles of $f(z_+)$ which show that $\gamma_0^{(n)} = f(0) < 1$ for all n . The plots also show that $\gamma_0^{(n)}$ decreases when the values of n increase: $\gamma_0^{(1)} = 0.398239039446192$, $\gamma_0^{(2)} = 0.0823639322747160$, $\gamma_0^{(3)} = 0.00673809869063151$, $\gamma_0^{(4)} = 0.000203456264553003$, $\gamma_0^{(5)} = 0.00000225915820929528$, The numerical analysis shows that $\gamma_0^{(n)} \rightarrow 0$ for large values of n . For all values of n and for large values of z_+ the profiles attain their asymptotic value 1.

The Fig. 2 shows the profiles of the gauge field $h(z_+)$. As expected, they are null in $r = 0$. The behavior, near the origin, for sufficiently large values of n , is linear in z_+ , in concordance with (35). For large values of z_+ , the profiles reach their asymptotic value n .

The Fig. 3 shows the profiles of the magnetic field $B(z_+)$. The profiles, for low values of n (here $n = 1, 2$),

FIG. 1: The scalar field profile $f(z_+)$.FIG. 3: The magnetic field $B(z_+)$.FIG. 2: The gauge field profile $f(z_+)$.FIG. 4: The BPS energy density $\varepsilon(z_+)$.

are lumps concentrated in the origin, whose amplitude is $|B(0)| = 1 - (\gamma_0)^2 < 1$. However, for sufficiently large values of n (here $n > 3$), the profiles develop a plateau whose width increases when n increases. Such behavior resembles that of the magnetic field of the Maxwell-Higgs vortices.

The Fig. 4 describes the BPS energy density profiles $\varepsilon(z_+)$. For $n = 1$ is a lump centered at origin. For sufficiently large values of n (here $n > 2$), the profiles form a plateau, starting at origin, with amplitude 1 whose width increases when n increases. At the ending of the plateau the profile increases its values reaching a maximum value. After this, the profiles decreases rapidly to zero. Such behavior also is shown in the BPS energy density of the Maxwell-Higgs vortices.

In summary we study the abelian Maxwell-Higgs model on an infinite cylinder surface. We explore the Bogomolnyi framework of the model in such surface, showing that it is possible to construct a topological soliton solutions. These solutions present interesting features, such as the behavior of the field ψ at $z_+ = 0$, which is different from zero, in contrast to Nielsen-Olesen vortex solution [1], where $\phi(0) = 0$. We also analyze carefully the numerical solutions for the matter and gauge fields and for the magnetic field and the energy density. Specifically, we have analyzed solutions for $n > 0$ however the solutions for $n < 0$ (anti-soliton solutions) can be easily visualized by doing: $h \rightarrow -h$, $f \rightarrow f$ and consequently $B \rightarrow -B$; it gives opposite magnetic flux, as expected for an anti-soliton solution. It is worthwhile to point out that there are also similar soliton solutions for $z < 0$.

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